# MINIMA OF DOUBLE INTEGRALS WITH RESPECT TO ONE-SIDED VARIATIONS.*

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In some problems in minima of double integrals the surface over which the integral is taken is restricted to lie in a given closed region $R$. Then it may happen that there is no extremal surface bounded by a previously given space curve which lies entirely in $R$, but that there is a surface bounded by the given curve, consisting of an extremal surface and a part of the boundary of $R$, which minimizes the given integral. In the first two sections of this paper some necessary conditions for such a minimum are derived, and in the last section there is a set of sufficient conditions.

The treatment of the analogous problem where the variations are unrestricted has been greatly simplified by Radon.† He shows that if the value of the double integral

$$\int\int \Phi\left(x, y, z, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) dudv$$

is to be unaffected by any change in the parametric representation of the surface over which the integral is taken, $\Phi$ must be expressible in the form

$$\Phi\left(x, y, z, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = F(x, y, z, A, B, C),$$

where

$$A = \frac{\partial(y, z)}{\partial(u, v)}, \qquad B = \frac{\partial(z, x)}{\partial(u, v)}, \qquad C = \frac{\partial(x, y)}{\partial(u, v)},$$

and then the proofs are much simpler. Consequently, the integral for which a minimum is sought will be given in the form

$$J = \int\int_{\Omega} F(x, y, z, A, B, C) dudv.$$

## § 1. The Analogue of the Lagrange Equation.

Suppose there is a surface of class $D''$‡ consisting of an extremal surface,

* Read before the American Mathematical Society, Apr. 24, 1915.

† Monatshefte für Mathematik und Physik, 22 (1911), p. 53.

‡ Bolza, Vorlesungen über Variationsrechnung, p. 664.

$$S: \qquad x = x(u, v), \quad y = y(u, v), \qquad z = z(u, v),$$

and a part of the boundary of $R$, whose equations are

$$\tilde{S}: \qquad x = \tilde{x}(u, v), \qquad y = \tilde{y}(u, v), \qquad z = \tilde{z}(u, v),$$

which gives as small a value to the integral $J$ as any other surface of class $D'$, belonging to $R$, that is in its neighborhood, and is bounded by the same space curve. The surface $\tilde{S}$ will be assumed to be of class $C''$, and its equations taken in such a way that $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ agree in sign with the direction cosines of the normal to $\tilde{S}$ directed into $R$. The intersection of $S$ and $\tilde{S}$ will be called

$$L: \qquad x = x(s), \qquad y = y(s), \qquad z = z(s),$$

where $s$ is the length of arc.

A function $\omega(u, v)$ will be selected, which is of class $D'$ in the neighborhood of $\tilde{S}$, vanishes along $L$, and is nowhere negative. Then if

$$\delta x = \epsilon\tilde{A}\omega(u, v), \qquad \delta y = \epsilon\tilde{B}\omega(u, v), \qquad \delta z = \epsilon\tilde{C}\omega(u, v),$$

the varied surface is entirely in $R$, for small positive values of $\epsilon$, and the first variation of $J$ becomes*

$$\delta J = \epsilon\int\int T\left(x, y, z, \frac{\partial x^2}{\partial u}, \cdots, \frac{\partial^2 z^2}{\partial v^2}\right)(\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2)w(u, v)dudv.$$

It follows that

$$T\left(x, y, z, \frac{\partial \tilde{x}^2}{\partial u}, \cdots, \frac{\partial \tilde{z}^2}{\partial v^2}\right) \geqq 0 \tag{1}$$

at every point of the part of the boundary of $R$ which belongs to the minimizing surface.

## § 2. The Angle of Intersection of $S$ and $\tilde{S}$.

It will next be proved that the surfaces $S$ and $\tilde{S}$ must intersect at such an angle that the Weierstrass $E$-function,

$$E(x, y, z; A, B, C; \tilde{A}, \tilde{B}, \tilde{C}) \equiv F(x, y, z, \tilde{A}, \tilde{B}, \tilde{C})$$
$$- \tilde{A}F_A(x, y, z, A, B, C) - \tilde{B}F_B(x, y, z, A, B, C) - \tilde{C}F_C(x, y, z, A, B, C),$$

vanishes at every point of $L$.

Suppose that there is a point $P$ where it does not vanish. Then there is a segment of $L$ including $P$ on which $E$ has a permanent sign. Let $\omega(s)$ be a function of class $C''$ which agrees in sign with $E$ in this segment and vanishes outside of it. The segment will be taken small enough to exclude any points where any of the derivatives $x'(s)$, $y'(s)$ and $z'(s)$ are

* Radon, loc. cit., p. 58, equation (15).

discontinuous. Then a part of $\tilde{S}$ including this segment of $L$ can be represented by equations of the form

$$x = \tilde{x}(s, n), \quad y = \tilde{y}(s, n), \quad z = \tilde{z}(s, n),$$

where $n$ is the length of a curve on $\tilde{S}$ normal to $L$. Since $E(x, y, z; A, B, C; \tilde{A}, \tilde{B}, \tilde{C})$ is homogeneous in $\tilde{A}, \tilde{B}, \tilde{C}$ such a change of parameters does not affect its non-vanishing. Three functions, $\xi(u, v)$, $\eta(u, v)$, $\zeta(u, v)$, will now be chosen in such a way that they vanish along the boundary of $\Omega$, and their values along $L$ are

$$\xi(u(s), v(s)) = \tilde{x}_n(s, 0)\omega(s),$$
$$\eta(u(s), v(s)) = \tilde{y}_n(s, 0)\omega(s),$$
$$\zeta(u(s), v(s)) = \tilde{z}_n(s, 0)\omega(s).$$

We take also the one-parameter family of surfaces defined by the equations

$$\bar{S}: \quad \begin{aligned} \bar{x} &= x(u, v) + \epsilon\xi(u, v), \\ \bar{y} &= y(u, v) + \epsilon\eta(u, v), \\ \bar{z} &= z(u, v) + \epsilon\zeta(u, v). \end{aligned}$$

The increment $\Delta J$ is then the difference between the value of the integral taken over $\bar{S}$ and a part of $\tilde{S}$, and its value taken over $S$ and a slightly different part of $\tilde{S}$. The first variation is found to be*

$$\begin{aligned} \delta J = \epsilon\int\int_S T(A\xi + B\eta + C\zeta)dudv - \epsilon\int_L \begin{vmatrix} F_A & F_B & F_C \\ \tilde{x}_n & \tilde{y}_n & \tilde{z}_n \\ \tilde{x}_s & \tilde{y}_s & \tilde{z}_s \end{vmatrix} \omega(s)ds \\ + \epsilon \lim_{\epsilon=0}\left[\frac{1}{\epsilon}\int\int_{\Delta\bar{S}} F(x, y, z, \bar{A}, \bar{B}, \bar{C})dudv \right. \\ \left. - \frac{1}{\epsilon}\int_L\int_0^{N(s, \epsilon)} F(x, y, z, \tilde{A}, \tilde{B}, \tilde{C})dnds\right]. \end{aligned} \tag{2}$$

The first integral vanishes because $S$ is an extremal surface. The region $\Delta\bar{S}$ is the part of $\bar{S}$ between its intersection with the set of tangents to $\tilde{S}$ normal to $L$, and its intersection with $\tilde{S}$ itself. Since the area of this region is of the second order with respect to $\epsilon$, the limit of the integral taken over it divided by $\epsilon$ is zero. The limit of the last integral is equal to the line integral

$$-\int_L F(x, y, z, \tilde{A}, \tilde{B}, \tilde{C})\omega(s)ds.$$

* Compare with Radon, loc. cit., p. 58, equation (15).

Since

$$\frac{\partial(\tilde{y}, \tilde{z})}{\partial(s, n)} = \tilde{A}, \quad \frac{\partial(\tilde{z}, \tilde{x})}{\partial(s, n)} = \tilde{B}, \quad \frac{\partial(\tilde{x}, \tilde{y})}{\partial(s, n)} = \tilde{C},$$

equation (2) may now be written

$$\delta J = -\epsilon \int_L E(x, y, z;\ A, B, C;\ \tilde{A}, \tilde{B}, \tilde{C})\omega(s)ds,$$

which does not vanish. Consequently there is no minimum. It follows that

$$E(x, y, z; A, B, C; \tilde{A}, \tilde{B}, \tilde{C}) = 0$$

at every point of $L$ is a necessary condition for a minimum.

This condition is satisfied identically if $S$ and $\tilde{S}$ have the same normal at each point of their intersection.

In the special case where $x$ and $y$ are the independent variables, and

$$J = \int\int f(x, y, z, p, q)dxdy,$$

the function $F$ becomes*

$$F(x, y, z, A, B, C) = f\left(x, y, z, \frac{-A}{C}, \frac{-B}{C}\right)C,$$

and $A = -p$, $B = -q$, $C = 1$. If these values are substituted in the function $T$,† it becomes

$$f_z - f_{px} - f_{qy} - f_{pz}p - f_{qz}q - f_{pp}\frac{\partial^2 z}{\partial x^2} - 2f_{pq}\frac{\partial^2 z}{\partial x \partial y} - f_{qq}\frac{\partial^2 z}{\partial y^2},$$

which is the left member of the Lagrange equation for the problem, and the Weierstrass $E$-function becomes

$$f(x, y, z, \tilde{p}, \tilde{q}) - f(x, y, z, p, q) - (\tilde{p} - p)f_p(x, y, z, p, q) - (\tilde{q} - q)f_q(x, y, z, p, q).$$

## § 3. Sufficient Conditions.

Sufficient conditions can be derived by a method similar to that used by Kneser‡ for the problem where the variations are unrestricted.

The first condition is that the given surface $S_0$ shall consist of an extremal surface, and a portion of $\tilde{S}$ the boundary of $R$, at every point of which the inequality (1) is satisfied, and the two parts of $S_0$ shall have the same normal at each point of their intersection. The second condition is that $S_0$ shall be embedded in a family of surfaces $S_a$, each consisting of an extremal surface and a part of $S_0$ or else of an extremal surface alone,

* Bolza, loc. cit., p. 665.

† Radon, loc. cit., p. 57.

‡ Kneser, Lehrbuch der Variationsrechnung, p. 300.

such that every point of $R$ in the neighborhood of $S_0$ lies on one and only one extremal surface of the family, and such that any of the extremal surfaces and $\tilde{S}$ shall have the same normal at each point of their intersection. The last condition is that if $A$, $B$, $C$ are proportional to the direction-cosines of the normal to the extremal surface and $\bar{A}$, $\bar{B}$, $\bar{C}$ proportional to the direction-cosines of any other line,

$$E(x, y, z; A, B, C; \bar{A}, \bar{B}, \bar{C}) > 0,$$

at every point in this neighborhood.

It will be proved that if these conditions are satisfied the surface $S_0$ gives a smaller value to the integral $J$ than any other surface $\bar{S}$ of class $D'$ which intersects $S_0$ along the boundary of the region over which the integral is taken, lies entirely in the neighborhood of $S_0$ defined above, and is not tangent to an infinite number of the surfaces $S_a$ along parts of their intersections.

It can be assumed that no part of $\bar{S}$ is between $S_0$ and $\tilde{S}$ because the variations are unrestricted over the part of $S_0$ which does not coincide with $\tilde{S}$, and sufficient conditions for a minimum with respect to unrestricted variations are assumed. The surface $\bar{S}$ will intersect a certain number of the surfaces $S_a$ in a set of closed curves $L_a$. The last surface $S_a$ which it touches will be called $S_{a_1}$. The function $J(a)$ will be defined as the integral $J$ taken over the part of $\bar{S}$ between $L_0$ and $L_a$ and over the part of $S_a$ bounded by $L_a$. Thus $J(0) = J(S_0)$ and $J(a_1) = J(\bar{S})$, and it must simply be proved that $J(a_1) - J(0)$ is positive.

If the equations of $S_a$ are

$$S_a: \qquad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

the equations of $S_{a+\Delta a}$ can be put in the form

$$S_{a+\Delta a}: \qquad \begin{aligned} x &= x(u, v) + A\omega(u, v, \Delta a), \\ y &= y(u, v) + B\omega(u, v, \Delta a), \\ z &= z(u, v) + C\omega(u, v, \Delta a). \end{aligned}$$

The parameter $a$ will be chosen in such a way that $a_1 > 0$ and the partial derivative $\omega_{\Delta a}(u, v, 0) > 0$ also. The equations of the part of $\bar{S}$ between $L_a$ and $L_{a+\Delta a}$, excepting possibly in the neighborhood of points where the direction angles of $L_a$ are discontinuous, can be put in the form

$$\Delta\bar{S}: \qquad x = \bar{x}(s, n), \qquad y = \bar{y}(s, n), \qquad z = \bar{z}(s, n),$$

where $s$ is the length of arc measured along $L_a$, and $n$ is the length of a curve normal to $L_a$. The value of $n$ on $L_{a+\Delta a}$ will be called $N(s, \cdot\, \Delta a)$.

The equations of the projection of $\Delta\bar{S}$ on $S_a$ will be expressed in terms of $s$ and $n'$, where $n'$ is the length of the projection of an $n$-curve. If $N'(s, \Delta a)$ is the value of $n'$ on the projection of $L_{a+\Delta a}$, and $\theta$ the angle between the normals to $\bar{S}$ and $S_a$, it is evident that the limit of $N'/N$ is $\cos\theta$, if $\Delta a$ approaches zero. The derivative of $J(a)$ is seen to be

$$\begin{aligned}\frac{dJ(a)}{da} &= \lim_{\Delta a=0}\int\int_{S_a}\frac{1}{\Delta a}(F(x+A\omega,\ y+B\omega,\ \cdots)-F(x,\ y,\ \cdots))dudv \\ &\quad+\int_{L_a}(F(x,\ y,\ z,\ \bar{A},\ \bar{B},\ \bar{C})-F(x,\ y,\ z,\ A,\ B,\ C)\cos\theta)N_{\Delta a}ds.\end{aligned}\tag{3}$$

The first integral is equal to

$$\int\int_{S_a}T(A^2+B^2+C^2)\omega_{\Delta a}dudv+\int_{L_a}\begin{vmatrix}F_A & F_B & F_C\\ A & B & C\\ \bar{x}_s(s,\ 0) & \bar{y}_s(s,\ 0) & \bar{z}_s(s,\ 0)\end{vmatrix}\omega_{\Delta a}ds.^*\tag{4}$$

The existence of the partial derivative $N_{\Delta a}$ can be proved by finding its value in terms of $\omega_{\Delta a}$. At every point of $L_{a+\Delta a}$

$$\begin{aligned}x(s,\ N')+A\omega(s,\ N',\ \Delta a) &= \bar{x}(s,\ N),\\ y(s,\ N')+B\omega(s,\ N',\ \Delta a) &= \bar{y}(s,\ N),\\ z(s,\ N')+C\omega(s,\ N',\ \Delta a) &= \bar{z}(s,\ N).\end{aligned}$$

If these equations are differentiated with respect to $\Delta a$ at $\Delta a = 0$, they become

$$\begin{aligned}A\omega_{\Delta a} &= (\bar{x}_n - x_{n'}\cos\theta)N_{\Delta a},\\ B\omega_{\Delta a} &= (\bar{y}_n - y_{n'}\cos\theta)N_{\Delta a},\\ C\omega_{\Delta a} &= (\bar{z}_n - z_{n'}\cos\theta)N_{\Delta a}.\end{aligned}$$

The coefficient of $N_{\Delta a}$ is different from zero in at least one of these equations unless $\bar{S}$ is tangent to $S_a$, and that case will be considered later. If these values are substituted for $A\omega_{\Delta a}$, $B\omega_{\Delta a}$ and $C\omega_{\Delta a}$ in the line integral in expression (4), it becomes

$$-\int_{L_a}[F_A(\bar{A}-A\cos\theta)+F_B(\bar{B}-B\cos\theta)+F_C(\bar{C}-C\cos\theta)]N_{\Delta a}ds.$$

Combining these results and making use of the identity

$$F(x,\ y,\ z,\ A,\ B,\ C) = AF_A+BF_B+CF_C,$$

* The sign of this line integral is different from the corresponding sign in equation (2), because the region considered is on the opposite side of the bounding curve.

we reduce equation (3) to

$$\frac{dJ(a)}{da} = \int\int_{S_a} T(A^2+B^2+C^2)\omega_{\Delta a} du dv + \int_{L_a} E(x, y, z; A, B, C; \bar{A}, \bar{B}, \bar{C}) N_{\Delta a} ds.$$

The double integral cannot be negative and the line integral is positive. Consequently

$$\frac{dJ(a)}{da} > 0.$$

If $S$ is tangent to $S_a$ along a part of one and only one curve $L_a$, such as $L_{a'}$, and $\delta$ is an arbitrarily small positive number, the inequalities

$$J(a_1) - J(a' + \delta) > 0, \qquad J(a' - \delta) - J(0) > 0,$$

are satisfied, and both quantities increase as $\delta$ decreases. But $|J(a' + \delta) - J(a' - \delta)|$ can be made arbitrarily small by taking $\delta$ sufficiently small. It is evident then that

$$J(\bar{S}) - J(S_0) = J(a_1) - J(a) > 0.$$

A similar argument can be made if $\bar{S}$ is tangent to a finite number of the surfaces $S_a$. The case where it is tangent to an infinite number of them will not be considered here.

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